Math 213, Fall 2025, Exam 1 SOLUTIONS
Read Carefully, Write Neatly, Sketch Accurately,
Only simplify as needed (e.g, you may leave messy square roots)

1. (42 points) For the points $P(2,1,1)$, $Q(4,2,3)$ and $R(6,4,1)$
   a. (4 points) Find the parametric equations of the line segment connecting P and Q.
   Solution:
   $\overrightarrow{PQ} = <4-2, 2-1, 3-1> = <2,1,2>$
   $\Rightarrow x = 2+2t; y = 1+t; z = 1+2t \quad 0 \le t \le 1$
   b. (4 points) Find the distance between the points P and Q.
   Solution:
   Distance is length of $\overrightarrow{PQ} = \sqrt{2^2+1^2+2^2} = 3$
   c. (4 points) Find the unit vector in the direction $\overrightarrow{PQ}$
   Solution: So the unit vector desired is $<2/3, 1/3, 2/3>$
   d. (8 points) Find the area of the triangle with vertices *P, Q,* and *R.*
   Solution: Area is half the magnitude of the cross product
   $\overrightarrow{PQ} = <2,1,2>; \overrightarrow{QR} = <2,2,-2>; \overrightarrow{RP} = <-4,-3,0>$
   $$\text{Area} = \frac{1}{2}\begin{vmatrix} i & j & k \\ 2 & 1 & 2 \\ 2 & 2 & -2 \end{vmatrix} = \frac{1}{2}\left|-6\mathbf{i}+8\mathbf{j}+2\mathbf{k}\right| = \frac{1}{2}\sqrt{36+64+4} = \frac{1}{2}\sqrt{104} = \sqrt{26}$$
   e. (8 points) Find the projection of $\overrightarrow{PQ}$ onto $\overrightarrow{PR}$
   Solution: The projections of $\overrightarrow{PQ}$ onto $\overrightarrow{PR}$ is given by
   $$\frac{\overrightarrow{PQ}\cdot\overrightarrow{PR}}{\left|\overrightarrow{PR}\right|^2}\overrightarrow{PR} = \frac{<2,1,2>\cdot<4,3,0>}{\sqrt{4^2+3^2}}\overrightarrow{PR} = \frac{11}{25}<4,3,0>$$
   f. (4 points) Find the equation of the plane in which the 3 points *P, Q,* and *R* lie.
   Solution: We already have the cross product, which gave the normal to the plane of the triangle, and is $-6\mathbf{i}+8\mathbf{j}+2\mathbf{k}$ so the equation of the plane is (using *P*, but one could use *Q* or *R*) $-6(x-2)+8(y-1)+2(z-1)=0$
   g. (10 points) Find the distance from the point (6,3,2) to this plane.
   Solution: To find the distance from the point to this plane find a vector from the point, *S*, to the plane, say (using *P*) $<6-2, 3-1, 2-1> = <4,2,1>$ and find the length of the projection onto the normal to the plane $-6\mathbf{i}+8\mathbf{j}+2\mathbf{k}$ given by
   $$\left|\frac{\overrightarrow{PS}}{\left|\overrightarrow{N}\right|}\cdot\overrightarrow{N}\right| = \left|\frac{<4,2,1>\cdot<-6,8,2>}{\sqrt{104}}\right| = \frac{6}{\sqrt{104}} = \frac{3}{\sqrt{26}}$$

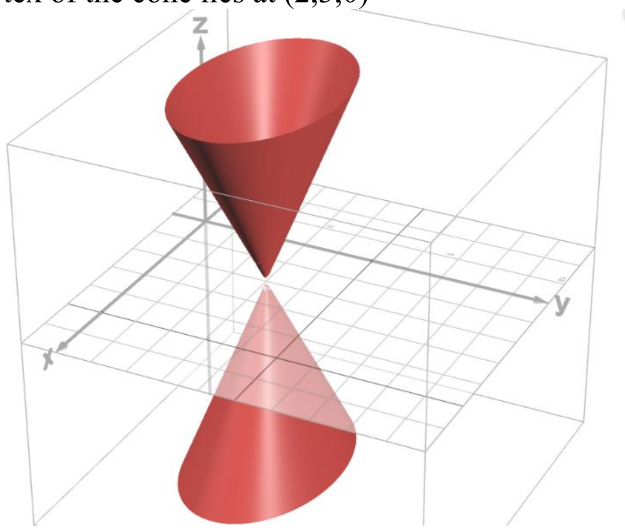
2. (8 points) For the equation $x^2+2y^2-4x-12y-z^2+22=0$, identify and sketch the surface. Explain cross sections perpendicular to any axes that might help clarify understanding your sketch.

Solution: Complete the square to get

$$x^2-4x+4-4+2(y-3)^2-18-z^2+22=0$$

$$\Rightarrow (x-2)^2+2(y-3)^2=z^2$$

This is an elliptical cone with cross sections parallel to the xy-plane being ellipses. The vertex of the cone lies at (2,3,0)



(from desmos)

3. (32 points) Let C be a curve parametrized by some $\mathbf{r}(t)$ where its velocity vector is given by $\mathbf{v}(t)=\mathbf{i}+t\sqrt{2}\mathbf{j}+t^2\mathbf{k}$ and the curve passes through the point $\mathbf{r}(3)=\mathbf{i}+4\sqrt{2}\mathbf{j}+5\mathbf{k}$
   a. (6 points) Find the curve's path, $\mathbf{r}(t)$.
   b. (4 points) Find the parametric equations for the line tangent to C at the point where $t = 3$.
   c. (8 points) Find the length of the curve for $t \in [0,3]$.
   d. (14 points) Find the curvature κ at $t = 3$

Solution:

a.

$$\mathbf{r}(t)=\mathbf{r}(3)+\int_3^t \mathbf{v}(t)\,dt=\mathbf{i}+4\sqrt{2}\mathbf{j}+5\mathbf{k}+\left(t\mathbf{i}+\frac{t^2}{2}\sqrt{2}\mathbf{j}+\frac{t^3}{3}\mathbf{k}\right)\Bigg|_3^t$$

$$\mathbf{r}(t)=(1+t-3)\mathbf{i}+\left(4\sqrt{2}+\frac{t^2}{2}\sqrt{2}-\frac{9}{2}\sqrt{2}\right)\mathbf{j}+\left(5+\frac{t^3}{3}-9\right)\mathbf{k}$$

$$=(t-2)\mathbf{i}+\left(\frac{t^2}{2}\sqrt{2}-\frac{1}{2}\sqrt{2}\right)\mathbf{j}+\left(\frac{t^3}{3}-4\right)\mathbf{k}$$

b. $\mathbf{v}(3)=\mathbf{i}+3\sqrt{2}\mathbf{j}+9\mathbf{k}$ so the tangent line is

$$\mathbf{r}(3)+t\mathbf{v}(3)=\mathbf{i}+4\sqrt{2}\mathbf{j}+5\mathbf{k}+t\left(\mathbf{i}+3\sqrt{2}\mathbf{j}+9\mathbf{k}\right)\Rightarrow$$

$$x=1+t;\, y=4\sqrt{2}+t3\sqrt{2};\, z=5+9t$$

c. The length of the curve is: (where $\left|\mathbf{v}(t)\right|=\sqrt{1^2+\left(t\sqrt{2}\right)^2+\left(t^2\right)^2}=\sqrt{1+2t^2+t^4}=1+t^2$)

$$\int_0^3\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt=\int_0^3\left|\mathbf{v}(t)\right|dt=\int_0^3\left(1+t^2\right)dt=\left(t+\frac{t^3}{3}\right)\Bigg|_0^3=3+9-0=12$$

d. For the curvature, we need

$$\mathbf{T}=\frac{\mathbf{v}}{|\mathbf{v}|}=\frac{\mathbf{i}+t\sqrt{2}\mathbf{j}+t^2\mathbf{k}}{1+t^2}\Rightarrow$$

$$\frac{d\mathbf{T}}{dt}=\frac{\mathbf{i}+t\sqrt{2}\mathbf{j}+t^2\mathbf{k}}{1+t^2}=\mathbf{i}\frac{-2t}{\left(1+t^2\right)^2}+\mathbf{j}\frac{\left(1+t^2\right)\sqrt{2}-2t^2\sqrt{2}}{\left(1+t^2\right)^2}+\mathbf{k}\frac{\left(1+t^2\right)2t-2t^3}{\left(1+t^2\right)^2}$$

$$\Rightarrow\frac{d\mathbf{T}}{dt}=\mathbf{i}\frac{-2t}{\left(1+t^2\right)^2}+\mathbf{j}\frac{\sqrt{2}-t^2\sqrt{2}}{\left(1+t^2\right)^2}+\mathbf{k}\frac{2t}{\left(1+t^2\right)^2}\Rightarrow\frac{d\mathbf{T}}{dt}_{t=3}=\mathbf{i}\frac{-6}{100}+\mathbf{j}\frac{-8\sqrt{2}}{100^2}+\mathbf{k}\frac{6}{100}$$

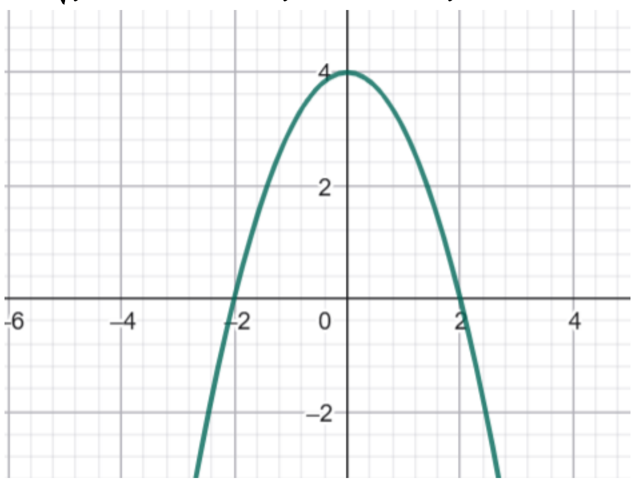
Also $\left|ds/dt\right|_{t=3}=\left|\mathbf{v}\right|_{t=3}=\left(1+t^2\right)_{t=3}=10$

$$\Rightarrow\kappa_{t=3}=\frac{\left|d\mathbf{T}/dt\right|_{t=3}}{\left|ds/dt\right|_{t=3}}=\frac{\frac{1}{100}\sqrt{6^2+\left(8\sqrt{2}\right)^2+6^2}}{\left|\mathbf{v}\right|_{t=3}}=\frac{\sqrt{200}}{1000}=\frac{10\sqrt{2}}{1000}=\frac{\sqrt{2}}{100}$$

4. (10 points) Find an equation for, and sketch the graph of, the level curve of the function $f(x,y)=\sqrt{y+x^2-3}$ that passes through the point (-1, 3).

Solution: Note that $f(-1,3)=\sqrt{3+1-3}=1$ so the level curve of interest is

$1=\sqrt{y+x^2-3}\Rightarrow 1=y+x^2-3\Rightarrow y=4-x^2$



5. (8 points) Use vector algebra to show that a parallelogram with diagonals of equal lengths is a rectangle. (Hint: Let the vectors representing the sides be **u** and **v.** Express the diagonals in terms of **u** and **v** and derive the consequence of the diagonals having equal lengths).

Solution: The diagonals of a parallelogram are $\mathbf{u}+\mathbf{v}$ and $-\mathbf{u}+\mathbf{v}$. Now,

$|\mathbf{u}+\mathbf{v}|=|-\mathbf{u}+\mathbf{v}|\Rightarrow$

$|\mathbf{u}+\mathbf{v}|^2=|-\mathbf{u}+\mathbf{v}|^2\Rightarrow$

$(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v})=(-\mathbf{u}+\mathbf{v})\cdot(-\mathbf{u}+\mathbf{v})\Rightarrow$

$|\mathbf{u}|^2+2\mathbf{u}\cdot\mathbf{v}+|\mathbf{v}|^2=|\mathbf{u}|^2-2\mathbf{u}\cdot\mathbf{v}+|\mathbf{v}|^2\Rightarrow$

$4\mathbf{u}\cdot\mathbf{v}=0\Rightarrow\mathbf{u}\cdot\mathbf{v}\Rightarrow\mathbf{u}\perp\mathbf{v}$

And so the parallelogram is a rectangle.